

## **A PRIMITIVE ASSOCIATED TO THE CANTOR-BENDIXSON DERIVATIVE ON THE REAL LINE**

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### **Abstract**

We consider the class of compact countable subsets of the real numbers  $\mathbb{R}$ . By using an appropriate partition, up to homeomorphism, of this class we give a detailed proof of a result shown by Mazurkiewicz and Sierpinski related to the cardinality of this partition. Furthermore, for any compact subset of  $\mathbb{R}$ , we show the existence of a “primitive” related to its Cantor-Bendixson derivative.

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## 1. Introduction

The earliest ideas of limit point and derived set in the space of the real numbers were both introduced and investigated by Georg Cantor since 1872 (see also [1, 2, 3, 4, 6]) to analyze the convergence set of a trigonometric series. These two concepts have been generalized to the case of any arbitrary topological space. Thus, let  $X$  be a topological space and let  $A$  be a subset of  $X$ , we write  $A'$  to denote the derived set of  $A$ , that is, the set of all limit points of  $A$ . The next definition extends the process of taking the derivative of a set for any ordinal number.

**Definition 1.1** (Cantor-Bendixson's derivative). Let  $A$  be a subset of a topological space. For a given ordinal number  $\alpha$ , we define, using transfinite recursion, the  $\alpha$ -th derivative of  $A$ , written  $A^{(\alpha)}$ , as follows:

- $A^{(0)} = A$ ,
- $A^{(\beta+1)} = (A^{(\beta)})'$ , for all ordinal  $\beta$ ,
- $A^{(\lambda)} = \bigcap_{\gamma < \lambda} A^{(\gamma)}$ , for all limit ordinal  $\lambda \neq 0$ .

In this paper, we are initially concerned with the Cantor-Bendixson derivative of compact countable subsets of the real numbers, where a countable set is either a finite set or a countably infinite set. Thus, we consider the set

$$\mathcal{K} = \{K \subset \mathbb{R} : K \text{ is compact and countable}\}. \quad (1.1)$$

Moreover, for all  $K_1, K_2 \in \mathcal{K}$ , we define the relation

$$K_1 \sim K_2 \Leftrightarrow \text{there exists } f : K_1 \mapsto K_2 \text{ continuous and bijective}. \quad (1.2)$$

It is not hard to see that  $\sim$  is an equivalence relation on the set  $\mathcal{K}$  and since the elements of  $\mathcal{K}$  are compact sets, we have that for all  $K_1, K_2 \in \mathcal{K}$

$$K_1 \sim K_2 \Leftrightarrow \text{there exists } f : K_1 \mapsto K_2 \text{ homeomorphism}. \quad (1.3)$$

Therefore, there is a partition of the set  $\mathcal{K}$ , and we denote by

$$\mathcal{X} = \mathcal{K} / \sim \tag{1.4}$$

the set of all equivalence classes of  $\mathcal{K}$ .

In 1920, Mazurkiewicz and Sierpinski [7] showed that the cardinality of  $\mathcal{X}$  is  $\aleph_1$ . In Section 2, we show in detail that for any countable ordinal number  $\alpha$ , and for any  $p \in \omega$ , there is a set  $K \in \mathcal{K}$  such that  $K^{(\alpha)}$  has exactly  $p$  elements. This last fact was first briefly mentioned by Cantor in [3]. The results shown in Section 2 allow us to prove, in Theorem 3.4, that the cardinality of  $\mathcal{X}$  is greater than or equal to  $\aleph_1$ . On the other hand, the cardinality of  $\mathcal{X}$  is smaller than or equal to  $\aleph_1$  as a consequence of Theorem 3.3.

Section 3 considers Cantor-Bendixson's characteristic, denoted by  $\mathcal{CB}$ . First, we show that for any element  $K \in \mathcal{K}$  with  $\mathcal{CB}(K) = (\alpha, p)$ , we get  $p = 0$  if and only if  $K = \emptyset$ . Moreover, we use Lemma 3.6 to prove Theorem 3.3, where the injectivity of function  $\widetilde{\mathcal{CB}}$ , defined in (3.12), is shown. These two last results were first mentioned in [7]; however, for the sake of completeness, we include here their detailed proofs. Finally, Theorem 3.5 shows that for any compact subset of the reals, there exists a primitive-like set connected with its Cantor-Bendixson derivative.

We recall that if  $F$  is a closed subset of  $\mathbb{R}$ , then  $(F^{(\alpha)})_{\alpha \in \mathbf{OR}}$  is a decreasing family of closed subsets of the real line. Furthermore, if  $K \in \mathcal{K}$ , then  $(K^{(\alpha)})_{\alpha \in \mathbf{OR}}$  is a decreasing family of elements of  $\mathcal{K}$ .

We denote by  $\mathbf{OR}$ , the class of all ordinal numbers. Moreover,  $\omega$  is used to designate the set of all natural numbers and  $\Omega$  represents the set of all countable ordinal numbers. In addition, the cardinality of a set  $B$  is denoted by  $|B|$ .

## 2. A Family of Elements in $\mathcal{K}$ Having a Cantor-Bendixson's Derivative with any Given Finite Number of Elements

First, we remark that any finite subset of  $\mathbb{R}$  is an element of  $\mathcal{K}$  with empty derived set. Thus, a set of this kind satisfies the property that its Cantor-Bendixson's derivative is empty for all ordinal number greater than or equal to 1. The following theorem let us find some elements belonging to  $\mathcal{K}$  not satisfying this last property. The main idea of the next result was given in [3], for completeness, we present below its proof in detail.

**Theorem 2.1.** *For any countable ordinal number  $\alpha \in \Omega$ , and for all  $a, b \in \mathbb{R}$  such that  $a < b$ , there is a set  $K \in \mathcal{K}$  such that  $K \subset (a, b]$  and  $K^{(\alpha)} = \{b\}$ .*

**Proof.** We will use transfinite induction.

(a) First, we consider the case  $\alpha = 0$ . For any  $a, b \in \mathbb{R}$  such that  $a < b$ , the result follows by taking the set  $K = \{b\} \in \mathcal{K}$ .

(b) Now, we suppose that for a given countable ordinal number  $\alpha \in \Omega$ , and for all  $c, d \in \mathbb{R}$  such that  $c < d$ , there is a set  $\tilde{K} \in \mathcal{K}$  such that  $\tilde{K} \subset (c, d]$  and  $\tilde{K}^{(\alpha)} = \{d\}$ . Let  $a, b \in \mathbb{R}$  be such that  $a < b$ . We take a strictly increasing sequence,  $(x_n)_{n \in \omega}$ , in  $(a, b]$  such that  $x_n \rightarrow b$  as  $n \rightarrow +\infty$ . Defining  $x_{-1} := a$  and applying the hypothesis to the real numbers  $x_{m-1} < x_m$ ,  $m \in \omega$ , it follows that there exists a sequence of sets  $(K_m)_{m \in \omega}$  such that for all  $m \in \omega$ ,  $K_m \in \mathcal{K}$ ,  $K_m \subset (x_{m-1}, x_m]$  and  $K_m^{(\alpha)} = \{x_m\}$ . Now, we define the set

$$K := \biguplus_{m \in \omega} K_m \uplus \{b\}. \quad (2.1)$$

The set  $K$ , given in (2.1), satisfies the following properties:

- $K \subset (a, b]$ , since  $K_m \subset (x_{m-1}, x_m] \subset (a, b]$ , for all  $m \in \omega$ .
- $K$  is countable, since it is the countable union of countable sets.
- $K$  is compact. In fact, given  $(A_i)_{i \in I}$  an open cover of  $K$ , there is a  $j \in I$  such that  $b \in A_j$ . Since  $A_j$  is an open set and  $(x_n)_{n \in \omega}$  is a strictly increasing sequence that converges to  $b$ , there exists  $N_1 \in \omega$  such that  $K_n \subset A_j$  for all  $n \in \omega$  with  $n > N_1$ . On the other hand, the set  $C := \bigcup_{n=0}^{N_1} K_n$  is compact, since it is the finite union of compact sets. Thus,  $C$  has a finite open subcover  $(A_i)_{i \in J}$ . Then,  $(A_i)_{i \in J \cup \{j\}}$  is a finite open subcover of  $K$ .
- For all ordinal number  $\beta$  with  $\beta \leq \alpha$ ,

$$K^{(\beta)} = \bigcup_{m \in \omega} K_m^{(\beta)} \cup \{b\}. \quad (2.2)$$

Last expression is obtained by using transfinite induction on  $\beta$ . In fact, the case  $\beta = 0$  is immediate from (2.1). Now, we suppose that for a given ordinal number  $\beta < \alpha$ , (2.2) holds. Since  $\beta + 1 \leq \alpha$ , we have that  $K_m^{(\alpha)} \subset K^{(\alpha)} \subset K^{(\beta+1)}$ , for all  $m \in \omega$ . Moreover, since  $x_m \in K_m^{(\alpha)} \subset K^{(\beta+1)}$ , for all  $m \in \omega$ , and  $x_m \rightarrow b$  as  $m \rightarrow +\infty$ , we see that  $b \in K^{(\beta+1)}$ . Therefore,

$$\bigcup_{m \in \omega} K_m^{(\beta+1)} \cup \{b\} \subset K^{(\beta+1)}. \quad (2.3)$$

In order to prove the other inclusion, let  $x \in K^{(\beta+1)}$ . Using the induction hypothesis, we see that

$$K^{(\beta+1)} \subset K^{(\beta)} = \bigcup_{m \in \omega} K_m^{(\beta)} \cup \{b\}.$$

Therefore, either  $x = b$  or  $x \in K_m^{(\beta)}$  for some  $m \in \omega$ . If  $x = b$ , then there is nothing else to prove. If  $x \neq b$ , there exists  $M \in \omega$  such that

$$x \in K_M^{(\beta)} \subset K_M \subset (x_{M-1}, x_M].$$

We claim that  $x \in K_M^{(\beta+1)}$ . To prove the last assertion, we suppose, by contradiction, that  $x \notin K_M^{(\beta+1)}$ . Thus,  $x$  is an isolated point of  $K_M^{(\beta)}$ . However, we know that  $\{x_M\} = K_M^{(\alpha)} \subset K_M^{(\beta+1)}$ . Then,  $x \neq x_M$ . Thus, there exists  $\varepsilon > 0$  such that  $(x - \varepsilon, x + \varepsilon) \subset (x_{M-1}, x_M)$  and

$$(x - \varepsilon, x + \varepsilon) \cap K_M^{(\beta)} = \{x\}.$$

Moreover, since  $(x - \varepsilon, x + \varepsilon) \subset (x_{M-1}, x_M)$ , we conclude that for all  $m \in \omega \setminus \{M\}$ ,

$$(x - \varepsilon, x + \varepsilon) \cap K_m^{(\beta)} = \emptyset.$$

Hence,

$$\begin{aligned} \{x\} &= (x - \varepsilon, x + \varepsilon) \cap \left( \biguplus_{m \in \omega} K_m^{(\beta)} \uplus \{b\} \right) \\ &= (x - \varepsilon, x + \varepsilon) \cap K^{(\beta)}, \end{aligned}$$

where in the last equality we have used the assumption that (2.2) holds for  $\beta$ . Even so, this last expression is a contradiction with the fact that  $x \in K^{(\beta+1)}$ . Then,  $x \in K_M^{(\beta+1)}$ . Thus,

$$K^{(\beta+1)} \subset \biguplus_{m \in \omega} K_m^{(\beta+1)} \uplus \{b\}. \quad (2.4)$$

Using (2.3) and (2.4), we get

$$K^{(\beta+1)} = \biguplus_{m \in \omega} K_m^{(\beta+1)} \uplus \{b\}.$$

Finally, let  $\gamma \neq 0$  be a limit ordinal such that  $\gamma \leq \alpha$  and suppose that

$$K^{(\delta)} = \biguplus_{m \in \omega} K_m^{(\delta)} \uplus \{b\}, \quad (2.5)$$

for all ordinal number  $\delta$  such that  $\delta < \gamma$ . Following a similar procedure to the one performed above to obtain (2.3), we have that

$$\biguplus_{m \in \omega} K_m^{(\gamma)} \uplus \{b\} \subset K^{(\gamma)}. \quad (2.6)$$

To obtain the other inclusion, let  $x \in K^{(\gamma)}$ . Using the induction hypothesis (2.5), we see that

$$K^{(\gamma)} := \bigcap_{\delta < \gamma} K^{(\delta)} = \bigcap_{\delta < \gamma} \left( \biguplus_{m \in \omega} K_m^{(\delta)} \uplus \{b\} \right).$$

Then, either  $x = b$  or for all ordinal number  $\delta$  such that  $\delta < \gamma$ , there exists  $m \in \omega$  such that  $x \in K_m^{(\delta)}$ . If  $x = b$ , then there is nothing left to prove. If  $x \neq b$ , there exists  $M \in \omega$  such that  $x \in K_M^{(0)} = K_M \subset (x_{M-1}, x_M]$ . We claim now that for all ordinal number  $\delta$  such that  $\delta < \gamma$ ,  $x \in K_M^{(\delta)}$ . In fact, we suppose, by contradiction, that there is an ordinal number  $\delta_0$  with  $\delta_0 < \gamma$  and such that  $x \notin K_M^{(\delta_0)}$ . However, we know that there exists  $m_0 \in \omega$  with  $m_0 \neq M$  such that  $x \in K_{m_0}^{(\delta_0)} \subset K_{m_0} \subset (x_{m_0-1}, x_{m_0}]$ . Since  $m_0 \neq M$ , we get  $(x_{m_0-1}, x_{m_0}] \cap (x_{M-1}, x_M] = \emptyset$ , which is a contradiction with the fact that  $x \in (x_{m_0-1}, x_{m_0}] \cap (x_{M-1}, x_M]$ . Therefore,

$$x \in \bigcap_{\delta < \gamma} K_M^{(\delta)} =: K_M^{(\gamma)} \subset \biguplus_{m \in \omega} K_m^{(\gamma)}.$$

Then,

$$K^{(\gamma)} \subset \biguplus_{m \in \omega} K_m^{(\gamma)} \uplus \{b\}. \quad (2.7)$$

By (2.6) and (2.7), we have that

$$K^{(\gamma)} = \biguplus_{m \in \omega} K_m^{(\gamma)} \uplus \{b\}.$$

Hence, (2.2) holds for all ordinal number  $\beta$  such that  $\beta \leq \alpha$ .

Applying now (2.2) to the ordinal number  $\alpha$ , and since  $K_m^{(\alpha)} = \{x_m\}$ , for all  $m \in \omega$ , we conclude that

$$\begin{aligned} K^{(\alpha)} &= \biguplus_{m \in \omega} K_m^{(\alpha)} \uplus \{b\} \\ &= \biguplus_{m \in \omega} \{x_m\} \uplus \{b\} \\ &= \{x_m : m \in \omega\} \uplus \{b\}. \end{aligned}$$

Therefore,

$$K^{(\alpha+1)} = (K^{(\alpha)})' = \{b\}.$$

(c) Finally, let  $\lambda \neq 0$  be a countable limit ordinal number. We suppose that for all ordinal number  $\rho$  such that  $\rho < \lambda$  and for all  $c, d \in \mathbb{R}$  such that  $c < d$ , there is a set  $\tilde{K} \in \mathcal{K}$  such that  $\tilde{K} \subset (c, d]$  and  $\tilde{K}^{(\rho)} = \{d\}$ . Since  $\lambda$  is a countable limit ordinal number, there exists a strictly increasing sequence  $(\rho_n)_{n \in \omega}$  in  $\Omega$  such that  $\rho_n < \lambda$ , for all  $n \in \omega$ , and  $\sup\{\rho_n : n \in \omega\} = \lambda$ . Let  $a, b \in \mathbb{R}$  be such that  $a < b$ . We take a strictly increasing sequence,  $(x_n)_{n \in \omega}$ , in  $(a, b]$  such that  $x_n \rightarrow b$  as  $n \rightarrow +\infty$ . Defining again  $x_{-1} = a$  and applying the hypothesis to the real numbers  $x_{m-1} < x_m$ , and the ordinal number  $\rho_m$ ,  $m \in \omega$ , it follows that there exists a sequence of sets  $(K_m)_{m \in \omega}$  such that for all  $m \in \omega$ ,  $K_m \in \mathcal{K}$ ,  $K_m \subset (x_{m-1}, x_m]$  and  $K_m^{(\rho_m)} = \{x_m\}$ . We also define, as in the previous case, the set

$$K := \biguplus_{m \in \omega} K_m \uplus \{b\}. \quad (2.8)$$

It can be shown, similarly to the case (b) above, that the set  $K$ , defined in (2.8), satisfies the following properties:

- $K \subset (a, b]$ .
- $K$  is countable.



- $K$  is compact.
- For all ordinal number  $\rho$  with  $\rho \leq \lambda$ ,

$$K^{(\rho)} = \biguplus_{m \in \omega} K_m^{(\rho)} \uplus \{b\}. \quad (2.9)$$

Last expression is obtained by using transfinite induction on  $\rho$ . In fact, the case  $\rho = 0$  is immediate from (2.8). Now, we suppose that for a given ordinal number  $\rho < \lambda$ , (2.9) holds. Since  $\lambda$  is a limit ordinal, we have that  $\rho + 1 < \lambda$ , and then there exists  $N \in \omega$  such that  $\rho + 1 < \rho_m$  for all  $m \in \omega$  with  $m > N$ . Therefore,  $x_m \in K_m^{(\rho_m)} \subset K_m^{(\rho+1)} \subset K^{(\rho+1)}$ , for all  $m \in \omega$  with  $m > N$ , and since  $x_m \rightarrow b$  as  $m \rightarrow +\infty$ , we see that  $b \in K^{(\rho+1)}$ . Then,

$$\biguplus_{m \in \omega} K_m^{(\rho+1)} \uplus \{b\} \subset K^{(\rho+1)}. \quad (2.10)$$

In order to prove the other inclusion, let  $x \in K^{(\rho+1)}$ . Using the induction hypothesis, we see that

$$K^{(\rho+1)} \subset K^{(\rho)} = \biguplus_{m \in \omega} K_m^{(\rho)} \uplus \{b\}.$$

Therefore, either  $x = b$  or  $x \in K_m^{(\rho)}$  for some  $m \in \omega$ . If  $x = b$ , then there is nothing else to prove. If  $x \neq b$ , there exists  $M \in \omega$  such that

$$x \in K_M^{(\rho)} \subset K_M \subset (x_{M-1}, x_M].$$

Since  $K_M^{(\rho_M+1)} = \emptyset$ , we have that  $\rho < \rho_M + 1$ , that is  $\rho \leq \rho_M$ . We claim that  $x \in K_M^{(\rho+1)}$ . To prove the last assertion, we suppose, by contradiction, that  $x \notin K_M^{(\rho+1)}$ . Thus,  $x$  is an isolated point of  $K_M^{(\rho)}$ . However, we know that  $K_M \cap K_{M+1} = \emptyset$ , then  $x \notin K_{M+1}$ . Hence,  $x \notin K_{M+1}^{(\rho)}$ . Thus, there exists  $\varepsilon > 0$  such that  $(x - \varepsilon, x + \varepsilon) \subset (x_{M-1}, x_{M+1})$ ,  $(x - \varepsilon, x + \varepsilon) \cap K_{M+1}^{(\rho)} = \emptyset$ , and

$$(x - \varepsilon, x + \varepsilon) \cap K_M^{(\rho)} = \{x\},$$

where in the second expression above we have used the fact that  $K_{M+1}^{(\rho)}$  is a closed subset of  $\mathbb{R}$ . Moreover, since  $(x - \varepsilon, x + \varepsilon) \subset (x_{M-1}, x_{M+1})$ , we conclude that for all  $m \in \omega \setminus \{M\}$ ,

$$(x - \varepsilon, x + \varepsilon) \cap K_m^{(\rho)} = \emptyset.$$

Hence,

$$\begin{aligned} \{x\} &= (x - \varepsilon, x + \varepsilon) \cap \left( \biguplus_{m \in \omega} K_m^{(\rho)} \uplus \{b\} \right) \\ &= (x - \varepsilon, x + \varepsilon) \cap K^{(\rho)}, \end{aligned}$$

where in the last equality we have used the assumption that (2.9) holds for  $\rho$ . Nevertheless, this last expression is a contradiction with the fact that  $x \in K^{(\rho+1)}$ . Then,  $x \in K_M^{(\rho+1)}$ . Thus,

$$K^{(\rho+1)} \subset \biguplus_{m \in \omega} K_m^{(\rho+1)} \uplus \{b\}. \quad (2.11)$$

Using (2.10) and (2.11), we get

$$K^{(\rho+1)} = \biguplus_{m \in \omega} K_m^{(\rho+1)} \uplus \{b\}.$$

Finally, let  $\gamma \neq 0$  be a limit ordinal such that  $\gamma \leq \lambda$  and suppose that

$$K^{(\delta)} = \biguplus_{m \in \omega} K_m^{(\delta)} \uplus \{b\}, \quad (2.12)$$

for all ordinal number  $\delta$  such that  $\delta < \gamma$ . We have, using (2.12), that

$$\begin{aligned} \biguplus_{m \in \omega} K_m^{(\gamma)} \uplus \{b\} &= \biguplus_{m \in \omega} \left( \bigcap_{\delta < \gamma} K_m^{(\delta)} \right) \uplus \{b\} \\ &\subset \bigcap_{\delta < \gamma} \left( \biguplus_{m \in \omega} K_m^{(\delta)} \right) \uplus \{b\} \end{aligned}$$

$$\begin{aligned}
&= \bigcap_{\delta < \gamma} \left( \biguplus_{m \in \omega} K_m^{(\delta)} \uplus \{b\} \right) \\
&= \bigcap_{\delta < \gamma} K^{(\delta)} \\
&= K^{(\gamma)}. \tag{2.13}
\end{aligned}$$

To get the other inclusion, we can follow a similar procedure to the one performed above to obtain (2.7). Thus, we have that

$$K^{(\gamma)} \subset \biguplus_{m \in \omega} K_m^{(\gamma)} \uplus \{b\}. \tag{2.14}$$

By (2.13) and (2.14), we obtain

$$K^{(\gamma)} = \biguplus_{m \in \omega} K_m^{(\gamma)} \uplus \{b\}.$$

Consequently, (2.9) holds for all ordinal number  $\rho$  such that  $\rho \leq \lambda$ . Furthermore, since for all  $m \in \omega$ ,  $\rho_m + 1 < \lambda$ , it follows that for all  $m \in \omega$

$$K_m^{(\lambda)} \subset K_m^{(\rho_m + 1)} = (K_m^{(\rho_m)})' = (\{x_m\})' = \emptyset.$$

Therefore,

$$K^{(\lambda)} = \biguplus_{m \in \omega} K_m^{(\lambda)} \uplus \{b\} = \{b\}.$$

From (a), (b), and (c), the theorem is proved.  $\square$

The next lemma will be used in the proof of Corollary 2.1 below.

**Lemma 2.1.** *Suppose that  $n \in \omega$ . Let  $F_1, F_2, \dots, F_n$  be closed subsets of  $\mathbb{R}$ . Then, for all ordinal number  $\alpha \in \mathbf{OR}$ , we have that*

$$\left( \bigcup_{k=1}^n F_k \right)^{(\alpha)} = \bigcup_{k=1}^n F_k^{(\alpha)}.$$

**Proof.** The general case,  $n \in \omega$ , is a consequence of the result for  $n = 2$  and the principle of finite induction. Thus, we suppose that  $n = 2$ . We will now use transfinite induction.

(a) If  $\alpha = 0$ , then there is nothing else to prove.

(b) We now suppose that for a given ordinal number  $\alpha \in \mathbf{OR}$ ,  $(F_1 \cup F_2)^{(\alpha)} = F_1^{(\alpha)} \cup F_2^{(\alpha)}$ . Therefore,

$$(F_1 \cup F_2)^{(\alpha+1)} = \left( (F_1 \cup F_2)^{(\alpha)} \right)' = \left( F_1^{(\alpha)} \cup F_2^{(\alpha)} \right)' = F_1^{(\alpha+1)} \cup F_2^{(\alpha+1)},$$

where in the last equation we have used the fact that the derived set of a finite union of subsets of a metric space equals the union of their derived sets.

(c) Finally, let  $\lambda \neq 0$  be a limit ordinal number. We suppose that for all  $\beta \in \mathbf{OR}$  such that  $\beta < \lambda$ ,  $(F_1 \cup F_2)^{(\beta)} = F_1^{(\beta)} \cup F_2^{(\beta)}$ . Then,

$$\begin{aligned} F_1^{(\lambda)} \cup F_2^{(\lambda)} &= \bigcap_{\beta < \lambda} F_1^{(\beta)} \cup \bigcap_{\beta < \lambda} F_2^{(\beta)} \\ &\subset \bigcap_{\beta < \lambda} (F_1^{(\beta)} \cup F_2^{(\beta)}) \\ &= \bigcap_{\beta < \lambda} (F_1 \cup F_2)^{(\beta)} \\ &= (F_1 \cup F_2)^{(\lambda)}. \end{aligned}$$

In order to prove the other inclusion, we take  $x \in (F_1 \cup F_2)^{(\lambda)}$ . We suppose, for the sake of contradiction, that  $x \notin F_1^{(\lambda)}$  and  $x \notin F_2^{(\lambda)}$ . Thus, there exist  $\beta_1, \beta_2 \in \mathbf{OR}$ , with  $\beta_1 < \lambda$  and  $\beta_2 < \lambda$ , such that  $x \notin F_1^{(\beta_1)}$  and  $x \notin F_2^{(\beta_2)}$ . If  $\beta_1 \leq \beta_2$ , then  $F_1^{(\beta_2)} \subset F_1^{(\beta_1)}$ . Hence,  $x \notin F_1^{(\beta_2)} \cup F_2^{(\beta_2)}$

$= (F_1 \cup F_2)^{(\beta_2)}$ , which contradicts the fact that  $x \in (F_1 \cup F_2)^{(\lambda)} = \bigcap_{\beta < \lambda} (F_1 \cup F_2)^{(\beta)}$ . The proof of the other case,  $\beta_2 < \beta_1$ , is similar. Therefore,

$$(F_1 \cup F_2)^{(\lambda)} = F_1^{(\lambda)} \cup F_2^{(\lambda)}.$$

Consequently, the lemma is proved.  $\square$

The following result is a generalization of Theorem 2.1.

**Corollary 2.1.** *Given any countable ordinal number  $\alpha$  and given any  $p \in \omega$ , there exists  $K \in \mathcal{K}$  such that  $|K^{(\alpha)}| = p$ .*

**Proof.** Let  $\alpha \in \Omega$ . If  $p = 0$ , we take  $K = \emptyset$ . If  $p \in \omega \setminus \{0\}$ , it is enough to apply Theorem 2.1 to a collection of  $p$  pairwise disjoint intervals. Thus, for all  $k \in \{1, \dots, p\}$ , there exists  $K_k \in \mathcal{K}$ , such that  $K_k^{(\alpha)}$  has only one element, and  $K_i \cap K_j = \emptyset$  for  $i, j \in \{1, \dots, p\}$  with  $i \neq j$ . We now define

$$K := \bigoplus_{k=1}^p K_k.$$

Hence,  $K \in \mathcal{K}$  and, using Lemma 2.1, we get

$$K^{(\alpha)} = \bigoplus_{k=1}^p K_k^{(\alpha)}.$$

Therefore,  $K^{(\alpha)}$  has exactly  $p$  elements.  $\square$

**Remark 2.1.** Even though the proofs of (2.2) and (2.9) are similar, it is worth mentioning that they are not identical. In fact, to prove (2.2) we have that  $\alpha \in \Omega$  and for all  $m \in \omega$ ,  $K_m^{(\alpha)} = \{x_m\}$ . On the other hand, to obtain (2.9), we consider  $\lambda \neq 0$  a countable limit ordinal and a strictly increasing sequence  $(\rho_m)_{m \in \omega}$  in  $\Omega$ , with  $\sup\{\rho_m : m \in \omega\} = \lambda$ , such that for all  $m \in \omega$ ,  $\rho_m < \lambda$  and  $K_m^{(\rho_m)} = \{x_m\}$ , where  $\rho_m$  depends on  $m$ .

In addition, we point out that the process developed to obtain (2.13) can also be used to get (2.3), (2.6), and (2.10).

### 3. Some Results Concerning Cantor-Bendixson's Derivative

It is a well-known fact that, for all  $K \in \mathcal{K}$ ,  $(K^{(\alpha)})_{\alpha \in \mathbf{OR}}$  is a decreasing family of elements of  $\mathcal{K}$ . The following two results were first proved by Cantor in [5] and they imply that for all  $K \in \mathcal{K}$ ,  $(K^{(\alpha)})_{\alpha \in \mathbf{OR}}$  is in fact a strictly decreasing family of sets in  $\mathcal{K}$  up to a countable ordinal number and such that all of its subsequent derivative sets are empty.

**Lemma 3.1.** *If  $K \in \mathcal{K}$  and  $K \neq \emptyset$ , then  $K' \neq K$ .*

The above lemma implies the following theorem.

**Theorem 3.1.** *If  $K \in \mathcal{K}$ , then there exists a countable ordinal number  $\beta$  such that  $K^{(\beta)}$  is finite.*

Since  $\Omega$  is a well-ordered set, by the previous theorem, we see that for all  $K \in \mathcal{K}$ , there exists the smallest countable ordinal number  $\alpha$  such that  $K^{(\alpha)}$  is finite. We can now give the next definition.

**Definition 3.1** (Cantor-Bendixson's characteristic). Let  $K \in \mathcal{K}$ . We say that  $(\alpha, p) \in \Omega \times \omega$  is the Cantor-Bendixson characteristic of  $K$  if  $\alpha$  is the smallest countable ordinal number such that  $K^{(\alpha)}$  is finite and  $|K^{(\alpha)}| = p$ . In this case, we write  $\mathcal{CB}(K) = (\alpha, p)$ .

By Theorem 2.1, for all countable ordinal number  $\alpha$ , there exists a set  $K \in \mathcal{K}$  having Cantor-Bendixson's characteristic  $(\alpha, 1)$ . Furthermore, by Corollary 2.1, we have that for all  $p \in \omega \setminus \{0\}$  and for all  $\alpha \in \Omega$ , there exists  $K \in \mathcal{K}$  such that  $\mathcal{CB}(K) = (\alpha, p)$ . In addition, we obviously see that  $\mathcal{CB}(\emptyset) = (0, 0)$ . Moreover, we have the next result concerning the empty set.

**Proposition 3.1.** *Let  $K \in \mathcal{K}$  be such that  $\mathcal{CB}(K) = (\alpha, p) \in \Omega \times \omega$ . Then,  $p = 0$  if and only if  $K = \emptyset$ .*

**Proof.** If  $K = \emptyset$ , then  $\mathcal{CB}(K) = (0, 0)$ , and thus the result holds. Now, we suppose that  $K \neq \emptyset$ . We consider three cases.

- If  $\alpha = 0$ , then  $K = K^{(0)}$  is finite. Since  $K \neq \emptyset$ , we have that  $|K^{(0)}| \neq 0$ . Hence,  $p \neq 0$ .

- We suppose now that  $\alpha$  is a nonzero limit ordinal. Then, for all  $\beta \in \Omega$  such that  $\beta < \alpha$ ,  $K^{(\beta)}$  is infinite. Therefore,  $(K^{(\beta)})_{\beta < \alpha}$  is a decreasing nested family of nonempty compact subsets of  $\mathbb{R}$ . By using the Cantor intersection theorem, we obtain

$$K^{(\alpha)} = \bigcap_{\beta < \alpha} K^{(\beta)} \neq \emptyset.$$

Then,  $|K^{(\alpha)}| \neq 0$ , and so  $p \neq 0$ .

- Finally, we assume that  $\alpha$  is a successor ordinal. Thus, there exists an ordinal  $\beta \in \Omega$  such that  $\beta + 1 = \alpha$ . Since  $\beta < \alpha$ , it follows that  $K^{(\beta)}$  is infinite. Then,

$$K^{(\alpha)} = K^{(\beta+1)} = (K^{(\beta)})' \neq \emptyset.$$

Therefore,  $|K^{(\alpha)}| \neq 0$ . Hence,  $p \neq 0$ . □

### 3.1. Partition of $\mathcal{K}$

In this subsection, we show some general results concerning the equivalence relation  $\sim$  defined on the set  $\mathcal{K}$  by (1.2).

**Proposition 3.2.** *Let  $K_1, K_2 \in \mathcal{K}$  be such that  $K_1 \sim K_2$ . Then,  $K'_1 \sim K'_2$ . More precisely, if  $f$  is a homeomorphism of  $K_1$  onto  $K_2$ , then  $f|_{K'_1}$  is also a homeomorphism of  $K'_1$  onto  $K'_2$ .*

**Proof.** Since the image of a limit point, under a homeomorphism, is also a limit point, we see that  $f(K'_1) = K'_2$ . Hence,  $f|_{K'_1} : K'_1 \mapsto K'_2$  is a homeomorphism. Therefore,  $K'_1 \sim K'_2$ .  $\square$

By using transfinite induction, we get the following result:

**Corollary 3.1.** *Let  $K_1, K_2 \in \mathcal{K}$  be such that  $K_1 \sim K_2$ , and let  $\alpha$  be any ordinal number. Then,  $K_1^{(\alpha)} \sim K_2^{(\alpha)}$ . More precisely, if  $f$  is a homeomorphism of  $K_1$  onto  $K_2$ , then  $f|_{K_1^{(\alpha)}}$  is also a homeomorphism of  $K_1^{(\alpha)}$  onto  $K_2^{(\alpha)}$ .*

It follows from the last corollary that if  $K_1, K_2 \in \mathcal{K}$ ,  $K_1 \sim K_2$  and  $\mathcal{CB}(K_1) = (\alpha, p) \in \Omega \times \omega$ , then there exists a bijective function of  $K_1^{(\alpha)}$  onto  $K_2^{(\alpha)}$ . Therefore,  $|K_2^{(\alpha)}| = |K_1^{(\alpha)}| = p$ . Hence,  $\mathcal{CB}(K_2) = (\alpha, p)$ . This last result about the Cantor-Bendixson characteristic, which was given by Mazurkiewicz and Sierpinski in [7], is expressed in the following theorem:

**Theorem 3.2.** *If  $K_1, K_2 \in \mathcal{K}$  and  $K_1 \sim K_2$ , then  $\mathcal{CB}(K_1) = \mathcal{CB}(K_2)$ .*

The above theorem shows that the Cantor-Bendixson characteristic is preserved for equivalent elements of  $\mathcal{K}$ , i.e., given  $K \in \mathcal{K}$ , we have that  $\mathcal{CB}(K_1) = \mathcal{CB}(K)$ , for all  $K_1 \in [K]$ , where  $[K]$  denotes the equivalence class of  $K$ . The reciprocal of Theorem 3.2, which was likewise given by Mazurkiewicz and Sierpinski in [7], is also true, and for completeness we give a more explicit proof of this fact in Theorem 3.3 below. In the following, we consider any ordinal number as a topological space with the order topology. Lemmas 3.2 to 3.6 will be used in the proof of Theorem 3.3.

**Lemma 3.2.** *Let  $K \in \mathcal{K}$  be such that  $\mathcal{CB}(K) = (1, 1)$ . Then, there exists a homeomorphism of  $K$  onto  $\omega + 1$ .*



**Proof.** There is an  $x \in \mathbb{R}$  such that  $K' = \{x\}$ . The set  $K \setminus K'$  is infinite and countable. Therefore, there exists a bijective function  $g$  of  $K \setminus K'$  onto  $\omega$ . Now, we define

$$f : K \mapsto \omega + 1$$

$$z \mapsto f(z) = \begin{cases} g(z), & \text{if } z \neq x, \\ \omega, & \text{if } z = x. \end{cases}$$

We see that  $f$  is a bijective function. Furthermore, since  $\omega + 1$  is a compact topological space,  $(\omega + 1)' = \{\omega\}$ ,  $f$  is an injective function, and  $f(K') = f(\{x\}) = \{\omega\}$ , we have that  $f$  is a continuous function. Moreover, since  $\omega + 1$  is a Hausdorff space, it follows that  $f$  is in fact a homeomorphism.  $\square$

**Lemma 3.3.** *Let  $\alpha$  be a countable ordinal number such that  $\alpha > 1$ . Suppose that for all ordinal number  $\beta$  such that  $0 < \beta < \alpha$  and for all  $\tilde{K} \in \mathcal{K}$  such that  $\mathcal{CB}(\tilde{K}) = (\beta, p) \in \Omega \times (\omega \setminus \{0\})$ , there exists a homeomorphism  $\tilde{f}$  of  $\tilde{K}$  onto  $\omega^\beta \cdot p + 1$ . Then, for all  $K \in \mathcal{K}$  such that  $\mathcal{CB}(K) = (\alpha, 1)$ , there exists a homeomorphism of  $K$  onto  $\omega^\alpha + 1$ .*

**Proof.** Let  $K \in \mathcal{K}$  be such that  $\mathcal{CB}(K) = (\alpha, 1)$ . Then, there exists an  $x \in K$  such that  $K^{(\alpha)} = \{x\}$ . We have that  $x \in K^{(\alpha)} \subset K''$ . Thus,  $x$  is a limit point of  $K'$ . Hence, there exists a strictly increasing or strictly decreasing sequence  $(x_n)_{n \in \omega}$  in  $K'$  such that it converges to  $x$ . We suppose that  $(x_n)_{n \in \omega}$  is an strictly increasing sequence in  $K'$ , the other case is similar.

We claim that for all  $n \in \omega$ , we can take  $r_n > 0$  such that  $x_n < x - r_n < x_{n+1}$  and  $x - r_n, x + r_n \notin K$ . In fact, if we suppose the contrary, then there exists  $l \in \omega$  such that

$$[x_l - x, x_{l+1} - x] \subset \{r \in \mathbb{R} : x - r \in K \text{ or } x + r \in K\}.$$

However, the set on the right-hand side of the last inclusion is countable, which is a contradiction. Hence, the claim is proved. We remark that the sequence  $(r_n)_{n \in \omega}$  converges to 0 as  $n$  goes to infinity. We now define the sets

$$\begin{aligned} K_0 &= K \cap ((-\infty, x - r_0] \cup [x + r_0, +\infty)), \\ K_k &= K \cap ([x - r_{k-1}, x - r_k] \cup [x + r_k, x + r_{k-1}]), \quad k \in \omega \setminus \{0\}. \end{aligned} \quad (3.1)$$

We see that for all  $k \in \omega$ ,  $x_k \in K_k$ . In addition, the sequence of sets  $(K_k)_{k \in \omega}$  satisfies the following properties:

- $K_k \subset K$ , for all  $k \in \omega$ .
- $K_k \in \mathcal{K}$ , for all  $k \in \omega$ , since they are countable closed subsets of  $K$ .
- $x_k \in K'_k \neq \emptyset$ , for all  $k \in \omega$ . In fact, let  $\varepsilon > 0$ . First, we consider the case  $k \in \omega \setminus \{0\}$ . We now take  $\hat{\varepsilon} := \min\{\varepsilon, x_k - x + r_{k-1}, x - r_k - x_k\} > 0$ . Since  $x_k \in K'$ , there exists  $z \in [(x_k - \hat{\varepsilon}, x_k + \hat{\varepsilon}) \setminus \{x_k\}] \cap K$ . Thus,  $z \in [(x_k - \varepsilon, x_k + \varepsilon) \setminus \{x_k\}] \cap K_k$ . Hence,  $x_k \in K'_k$ . For the case  $k = 0$ , by taking  $\hat{\varepsilon} := \min\{\varepsilon, x - r_0 - x_0\} > 0$ , and proceeding in a similar way as in the previous case, we see that  $x_0 \in K'_0$ .

- $(K_k)_{k \in \omega}$  is a pairwise disjoint sequence in  $\mathcal{K}$ .
- $\biguplus_{k \in \omega} K_k \uplus \{x\} = K$ . The fact that  $\biguplus_{k \in \omega} K_k \uplus \{x\} \subset K$  follows directly from (3.1). In order to prove the reverse inclusion, we take  $z \in K$ . If  $z = x$ , there is nothing else to show. Now, we suppose that  $z \neq x$ . Since  $r_n \rightarrow 0$  as  $n \rightarrow +\infty$ , we can choose the smallest natural number  $N \in \omega$  such that  $r_N < |x - z|$ . Then,  $z \in K_N$ .

Moreover, from (3.1), we see that for all  $k \in \omega$ ,  $x \notin K_k^{(\alpha)} \subset \{x\}$ . Therefore, for all  $k \in \omega$ ,  $K_k^{(\alpha)} = \emptyset$ . Thus, for all  $k \in \omega$ ,  $\mathcal{CB}(K_k) = (\beta_k, p_k) \in \Omega \times \omega$  implies that  $0 < \beta_k < \alpha$ . We remark that for all  $k \in \omega$ ,  $K_k \neq \emptyset$

implies that  $p_k \in \omega \setminus \{0\}$ . Using the hypothesis, we conclude that for all  $k \in \omega$ , there exists a homeomorphism  $f_k$  of  $K_k$  onto  $\omega^{\beta_k} \cdot p_k + 1$ . We now define the function

$$f : K \mapsto \tau + 1$$

$$z \mapsto f(z) = \begin{cases} f_0(z), & \text{if } z \in K_0, \\ \sum_{j=0}^{k-1} \omega^{\beta_j} \cdot p_j + 1 + f_k(z), & \text{if } z \in K_k, k \in \omega \setminus \{0\}, \\ \tau, & \text{if } z = x, \end{cases}$$

where

$$\tau := \sum_{k \in \omega} \omega^{\beta_k} \cdot p_k := \sup \left\{ \sum_{k=0}^n \omega^{\beta_k} \cdot p_k : n \in \omega \right\}.$$

(a) First, we remark that  $f$  is an injective function. In fact, let  $u, v \in K$  be such that  $f(u) = f(v)$ . If  $u = x$  and  $v \in K_q$ , for some  $q \in \omega$ , then  $f(v) \leq \sum_{k=0}^q \omega^{\beta_k} \cdot p_k < \tau = f(u)$ , which is a contradiction. Thus, there exists  $r \in \omega$  such that  $u \in K_r$ . We suppose, by contradiction, that  $q \neq r$ . Without loss of generality, we may assume that  $q < r$ . Then,

$$\begin{aligned} f(v) &\leq \sum_{k=0}^n \omega^{\beta_k} \cdot p_k \leq \sum_{k=0}^{r-1} \omega^{\beta_k} \cdot p_k \\ &< \sum_{k=0}^{r-1} \omega^{\beta_k} \cdot p_k + 1 + f_r(u) = f(u), \end{aligned}$$

which is not possible. Hence,  $q = r$ . Thus,

$$\sum_{k=0}^{q-1} \omega^{\beta_k} \cdot p_k + 1 + f_q(u) = f(u) = f(v) = \sum_{k=0}^{q-1} \omega^{\beta_k} \cdot p_k + 1 + f_q(v),$$

implies that  $f_q(u) = f_q(v)$ . Using the fact that  $f_q$  is an injective function, it follows that  $u = v$ .

(b) We will now show that  $f$  is onto. In fact, let  $\gamma \leq \tau$ . If  $\gamma = \tau$ , we have that  $f(x) = \tau = \gamma$ . If  $\gamma < \tau$ , we take  $M := \min\{n \in \omega : \gamma \leq \sum_{k=0}^n \omega^{\beta_k} \cdot p_k\}$ . In case  $M = 0$ ,  $\gamma \leq \omega^{\beta_0} \cdot p_0$ . Since,  $f_0$  is onto, there exists  $z \in K_0 \subset K$  such that  $f(z) = f_0(z) = \gamma$ . We now assume that  $M \in \omega \setminus \{0\}$ . Then,

$$\sum_{k=0}^{M-1} \omega^{\beta_k} \cdot p_k + 1 \leq \gamma \leq \sum_{k=0}^M \omega^{\beta_k} \cdot p_k.$$

Thus, there exists an ordinal number  $\mu$  such that

$$\sum_{k=0}^{M-1} \omega^{\beta_k} \cdot p_k + 1 + \mu = \gamma \leq \sum_{k=0}^{M-1} \omega^{\beta_k} \cdot p_k + \omega^{\beta_M} \cdot p_M.$$

Then,  $\mu \leq \omega^{\beta_M} \cdot p_M$ . Since  $f_M$  is onto, there exists  $z \in K_M \subset K$  such that  $f_M(z) = \mu$ . So,  $f(z) = \sum_{k=0}^{M-1} \omega^{\beta_k} \cdot p_k + 1 + f_M(z) = \gamma$ .

(c) Moreover, for all  $k \in \omega$ ,  $f|_{K_k}$  equals an ordinal number, i.e., a constant function, plus a continuous function. Thus, for all  $k \in \omega$ ,  $f|_{K_k}$  is a continuous function. In addition, since  $(K_k)_{k \in \omega}$  is a pairwise disjoint sequence of open subsets in  $K$ , it follows that  $f$  is a continuous function at any element of  $\biguplus_{k \in \omega} K_k$ . Furthermore,  $f$  is also continuous at the point  $x \in K$ . In fact, let  $\mu$  be an ordinal number such that  $\mu < \tau$ . There exists  $m \in \omega$  such that  $\mu < \sum_{j=0}^m \omega^{\beta_j} \cdot p_j$ . We claim that

$$f((x - r_m, x + r_m) \cap K) \subset (\mu, \tau + 1). \quad (3.2)$$

Let  $y \in (x - r_m, x + r_m) \cap K$ . If  $y = x$ , then  $f(y) = f(x) = \tau \in (\mu, \tau + 1)$ . We now suppose that  $y \neq x$ . Then, there is  $i \in \omega$  such that  $y \in K_i$ . Since  $(r_n)_{n \in \omega}$  is a strictly decreasing sequence of positive numbers, we conclude that  $i > m$ . Then,

$$f(y) = \sum_{j=0}^{i-1} \omega^{\beta_j} \cdot p_j + 1 + f_i(y) \geq \sum_{j=0}^m \omega^{\beta_j} \cdot p_j > \mu. \quad (3.3)$$

Moreover,

$$\begin{aligned} f(y) &= \sum_{j=0}^{i-1} \omega^{\beta_j} \cdot p_j + 1 + f_i(y) \leq \sum_{j=0}^{i-1} \omega^{\beta_j} \cdot p_j + 1 + \omega^{\beta_i} \cdot p_i \\ &= \sum_{j=0}^i \omega^{\beta_j} \cdot p_j \leq \tau < \tau + 1. \end{aligned} \quad (3.4)$$

From Equations (3.3) and (3.4), we see that  $f(y) \in (\mu, \tau + 1)$ . Thus, Equation (3.2) follows. Hence,  $f$  is continuous at the point  $x$ .

By (a) and (b),  $f$  is a bijective function. In addition, by (c),  $f$  is a continuous function of  $K$  onto  $\tau + 1$ .

We will now prove that  $\tau = \omega^\alpha$ . In order to get this, let  $\tilde{\alpha} := \sup\{\beta_k : k \in \omega\} \in \mathbf{OR}$ . We see that  $\tilde{\alpha} \leq \alpha$ .

(i) First, we consider the case  $\tilde{\alpha} < \alpha$ . Then,  $\tilde{\alpha} + 1 \leq \alpha$ . Thus, for all  $k \in \omega$ ,  $K_k^{(\tilde{\alpha}+1)} = \emptyset$ . Using transfinite induction, and proceeding as in the proof of (2.2), we get

$$K^{(\tilde{\alpha}+1)} = \bigsqcup_{k \in \omega} K_k^{(\tilde{\alpha}+1)} \sqcup \{x\} = \{x\}.$$

Then,  $\tilde{\alpha} + 1 = \alpha$ . Since for all  $k \in \omega$ ,  $\omega^{\beta_k} \cdot p_k \leq \omega^{\tilde{\alpha}} \cdot p_k$ , we see that

$$\tau = \sum_{k \in \omega} \omega^{\beta_k} \cdot p_k \leq \omega^{\tilde{\alpha}} \cdot \left( \sum_{k \in \omega} p_k \right) = \omega^{\tilde{\alpha}} \cdot \omega = \omega^{\tilde{\alpha}+1} = \omega^\alpha. \quad (3.5)$$

On the other hand, we claim that

$$|\{n \in \omega : \beta_n = \tilde{\alpha}\}| = \aleph_0. \quad (3.6)$$

In order to prove (3.6), we first suppose, by contradiction, that for all  $n \in \omega$ ,  $\beta_n < \tilde{\alpha}$ . Thus, for all  $n \in \omega$ ,  $\beta_n + 1 \leq \tilde{\alpha}$ , and we get  $K_n^{(\tilde{\alpha})} \subset$

$K_n^{(\beta_n+1)} = \emptyset$ . Moreover, we see that  $K^{(\tilde{\alpha})} = \biguplus_{k \in \omega} K_k^{(\tilde{\alpha})} \uplus \{x\} = \{x\}$ . Then,  $\tilde{\alpha} = \alpha$ , which is a contradiction. Hence, there exists at least one  $n \in \omega$  such that  $\beta_n = \tilde{\alpha}$ . We now suppose, again by contradiction, that the set  $\{n \in \omega : \beta_n = \tilde{\alpha}\} \neq \emptyset$  is finite. Let  $N := \max\{n \in \omega : \beta_n = \tilde{\alpha}\} \in \omega$ . We have that for all  $k \in \omega$  such that  $k > N$ ,  $\beta_k < \tilde{\alpha}$ . Then,

$$K^{(\tilde{\alpha})} = \biguplus_{k \in \omega} K_k^{(\tilde{\alpha})} \uplus \{x\} = \biguplus_{k=0}^N K_k^{(\tilde{\alpha})} \uplus \{x\}.$$

It follows that,  $K^{(\tilde{\alpha})}$  is a finite set. Hence,  $K^{(\alpha)} = K^{(\tilde{\alpha}+1)} = \emptyset$ , which is a contradiction with the fact that  $K^{(\alpha)} = \{x\}$ . Therefore, (3.6) is proved. We now define, for all  $n \in \omega$ ,

$$m_n := |\{k \in \omega : k \leq n \text{ and } \beta_k = \tilde{\alpha}\}| \in \omega.$$

Then, for all  $n \in \omega$ , we have that

$$\sum_{k=0}^n \omega^{\beta_k} \cdot p_k \geq \omega^{\tilde{\alpha}} \cdot m_n.$$

For this reason,

$$\begin{aligned} \tau &= \sum_{k \in \omega} \omega^{\beta_k} \cdot p_k \geq \omega^{\tilde{\alpha}} \cdot \sup\{m_n : n \in \omega\} \\ &= \omega^{\tilde{\alpha}} \cdot \omega = \omega^{\tilde{\alpha}+1} = \omega^{\alpha}. \end{aligned} \tag{3.7}$$

Using (3.5) and (3.7), we conclude that  $\tau = \omega^{\alpha}$ .

(ii) We now consider the case  $\tilde{\alpha} = \alpha$ . We claim that for all  $k \in \omega$ ,  $\beta_k < \tilde{\alpha}$ . In fact, if there exists  $l \in \omega$  such that  $\beta_l = \tilde{\alpha}$ , then

$$K_l^{(\beta_l)} \uplus \{x\} \subset \biguplus_{i \in \omega} K_i^{(\beta_l)} \uplus \{x\} = K^{(\beta_l)} = K^{(\alpha)} = \{x\},$$

contradicting the fact that  $|K_l^{(\beta_l)}| = p_l > 0$ . We now remark that  $\alpha$  is a limit ordinal. In order to prove the last assertion, we suppose, for the

sake of contradiction, that  $\alpha$  is a successor ordinal. Then, there exists an ordinal number  $\lambda$  such that  $\alpha = \lambda + 1$ . Thus, for all  $k \in \omega$ ,  $\beta_k \leq \lambda < \alpha = \tilde{\alpha}$ , which is a contradiction with the definition of  $\tilde{\alpha}$ . On the other hand, since for all  $k \in \omega$ ,  $\omega^{\beta_k} \leq \omega^{\beta_k} \cdot p_k \leq \tau$ , it follows that

$$\omega^\alpha = \omega^{\tilde{\alpha}} = \sup\{\omega^{\beta_k} : k \in \omega\} \leq \tau. \quad (3.8)$$

We now define, for all  $n \in \omega$ ,

$$\beta_{k_n} := \max\{\beta_k : k = 0, 1, \dots, n\},$$

$$p_{k_n} := \max\{p_k : k = 0, 1, \dots, n\}.$$

Then, for all  $n \in \omega$ , we see that

$$\sum_{k=0}^n \omega^{\beta_k} \cdot p_k \leq \omega^{\beta_{k_n}} \cdot p_{k_n} \cdot n < \omega^{\beta_{k_n} + 1} \leq \omega^\alpha,$$

where in the last inequality we have used the fact that  $\beta_{k_n} < \beta_{k_n} + 1 \leq \alpha$ .

In consequence,

$$\tau = \sum_{k \in \omega} \omega^{\beta_k} \cdot p_k \leq \omega^\alpha. \quad (3.9)$$

Equations (3.8) and (3.9) imply that  $\tau = \omega^\alpha$ .

Therefore,  $f$  is a bijective and continuous function of  $K$  onto  $\tau + 1 = \omega^\alpha + 1$ . In addition, since  $\omega^\alpha + 1$  is a Hausdorff space, we conclude that  $f$  is a homeomorphism of  $K$  onto  $\omega^\alpha + 1$ .  $\square$

**Lemma 3.4.** *Suppose that  $K$  and  $F$  are closed subsets of  $\mathbb{R}$  such that  $K \cap F = K \cap \overset{\circ}{F}$ , where  $\overset{\circ}{F}$  is the set of all interior points of  $F$ . Then, for all  $\alpha \in \mathbf{OR}$ , we have that*

$$(K \cap F)^{(\alpha)} = K^{(\alpha)} \cap F. \quad (3.10)$$

**Proof.** We proceed by transfinite induction.

- The case  $\alpha = 0$  is immediate.
- We now suppose that the result is true for  $\alpha \in \mathbf{OR}$ . Then,

$$(K \cap F)^{(\alpha+1)} = \left( (K \cap F)^{(\alpha)} \right)' = (K^{(\alpha)} \cap F)' \subset (K^{(\alpha)})' \cap F' \subset K^{(\alpha+1)} \cap F,$$

where in the last expression we have used the induction hypothesis and the fact that  $F$  is closed. In order to prove the reverse inclusion, let  $x \in K^{(\alpha+1)} \cap F$ . Since  $K$  is closed,  $x \in K \cap F = K \cap \overset{\circ}{F}$ . Thus, there exists  $r > 0$  such that  $(x - r, x + r) \subset F$ . Let  $\varepsilon > 0$ . We now take  $\tilde{\varepsilon} := \min\{\varepsilon, r\} > 0$ . Then,

$$\begin{aligned} \emptyset \neq ((x - \tilde{\varepsilon}, x + \tilde{\varepsilon}) \setminus \{x\}) \cap K^{(\alpha)} &= ((x - \tilde{\varepsilon}, x + \tilde{\varepsilon}) \setminus \{x\}) \cap K^{(\alpha)} \cap F \\ &\subset ((x - \varepsilon, x + \varepsilon) \setminus \{x\}) \cap (K \cap F)^{(\alpha)}. \end{aligned}$$

Hence,  $x \in (K \cap F)^{(\alpha+1)}$ . Therefore,  $(K \cap F)^{(\alpha+1)} = K^{(\alpha+1)} \cap F$ .

- Finally, let  $\lambda \neq 0$  be a limit ordinal number. We suppose that for all  $\beta \in \mathbf{OR}$  such that  $\beta < \lambda$ ,  $(K \cap F)^{(\beta)} = K^{(\beta)} \cap F$ . Then,

$$(K \cap F)^{(\lambda)} = \bigcap_{\beta < \lambda} (K \cap F)^{(\beta)} = \bigcap_{\beta < \lambda} (K^{(\beta)} \cap F) = \bigcap_{\beta < \lambda} K^{(\beta)} \cap F = K^{(\lambda)} \cap F.$$

This concludes the proof.  $\square$

**Lemma 3.5.** *Let  $\alpha$  be a countable ordinal number such that  $\alpha > 0$ . Let  $p \in \omega \setminus \{0\}$ . Suppose that for all  $\tilde{K} \in \mathcal{K}$  such that  $\mathcal{CB}(\tilde{K}) = (\alpha, 1)$ , there exists a homeomorphism of  $\tilde{K}$  onto  $\omega^\alpha + 1$ . Then, for all  $K \in \mathcal{K}$  such that  $\mathcal{CB}(K) = (\alpha, p)$ , there exists a homeomorphism of  $K$  onto  $\omega^\alpha \cdot p + 1$ .*



**Proof.** Let  $K \in \mathcal{K}$  be such that  $\mathcal{CB}(K) = (\alpha, p) \in \Omega \times \omega$ . We write  $K^{(\alpha)} = \{x_1, x_2, \dots, x_p\}$ , where  $x_i < x_j$ , for all  $i, j \in I := \{1, \dots, p\}$  with  $i < j$ . We see that for all  $k \in \{1, \dots, p-1\}$ , there exists  $z_k \in (x_k, x_{k+1})$  such that  $z_k \notin K$ . We now consider the sets

$$\begin{aligned} K_1 &= K \cap (-\infty, z_1], \\ K_k &= K \cap [z_{k-1}, z_k], \quad k \in \{2, \dots, p-1\}, \\ K_p &= K \cap [z_{p-1}, +\infty). \end{aligned} \tag{3.11}$$

Proceeding as in the proof of Lemma 3.3, it is possible to show that the finite family  $(K_k)_{k \in I}$  satisfies the following properties:

- $K_k \subset K$ , for all  $k \in I$ .
- $K_k \in \mathcal{K}$ , for all  $k \in I$ .
- $x_k \in K'_k \neq \emptyset$ , for all  $k \in I$ .
- $(K_k)_{k \in I}$  is a pairwise disjoint finite sequence in  $\mathcal{K}$ .
- $\biguplus_{k \in I} K_k = K$ .

By using Lemma 3.4, we have that for all  $k \in I$ ,  $K_k^{(\alpha)} = \{x_k\}$ . Therefore, for all  $k \in I$ ,  $\mathcal{CB}(K_k) = (\alpha, 1)$ . Thus, for all  $k \in I$ , there exists a homeomorphism  $f_k$  of  $K_k$  onto  $\omega^\alpha + 1$ . We now define the function  $f$  given by

$$f : K \mapsto \tau + 1$$

$$z \mapsto f(z) = \begin{cases} f_1(z), & \text{if } z \in K_1, \\ \sum_{j=1}^{k-1} \omega^\alpha + 1 + f_k(z), & \text{if } z \in K_k, \text{ for some } k \in I \setminus \{1\}, \end{cases}$$

where

$$\tau := \sum_{j=1}^p \omega^\alpha = \omega^\alpha \cdot \sum_{j=1}^p 1 = \omega^\alpha \cdot p.$$

Proceeding in a similar fashion as in the items (a), (b), and (c) in the proof of Lemma 3.3, we obtain that  $f$  is a homeomorphism of  $K$  onto  $\omega^\alpha \cdot p + 1$ .

□

**Lemma 3.6.** *Let  $\alpha$  be a countable ordinal number such that  $\alpha > 0$ . Let  $p \in \omega \setminus \{0\}$ . Then, for all  $K \in \mathcal{K}$  such that  $\mathcal{CB}(K) = (\alpha, p)$ , there exists a homeomorphism of  $K$  onto  $\omega^\alpha \cdot p + 1$ .*

**Proof.** We will use strong transfinite induction. By Lemmas 3.2 and 3.5, the result holds for  $\alpha = 1$ . We now consider  $\alpha \in \Omega$  such that  $\alpha > 1$ , and we suppose that the result is true for all ordinal number  $\beta$  such that  $0 < \beta < \alpha$ . Lemmas 3.3 and 3.5 imply the result for  $\alpha$ . Hence, the lemma is proved. □

Next result contains the reciprocal of Theorem 3.2.

**Theorem 3.3.** *If  $K_1, K_2 \in \mathcal{K}$  and  $\mathcal{CB}(K_1) = \mathcal{CB}(K_2)$ , then  $K_1 \sim K_2$ .*

**Proof.** If  $\mathcal{CB}(K_1) = \mathcal{CB}(K_2) = (0, p) \in \Omega \times \omega$ , we get  $|K_1| = |K_2| = p$ . Then,  $K_1 \sim K_2$ .

We now suppose that  $\mathcal{CB}(K_1) = \mathcal{CB}(K_2) = (\alpha, p)$ , with  $\alpha > 0$ . By Proposition 3.1,  $p \in \omega \setminus \{0\}$ . By Lemma 3.6, there exist two homeomorphisms,  $g$  of  $K_1$  onto  $\omega^\alpha \cdot p + 1$  and  $h$  of  $K_2$  onto  $\omega^\alpha \cdot p + 1$ . Therefore,  $f = h^{-1} \circ g : K_1 \mapsto K_2$  is a homeomorphism of  $K_1$  onto  $K_2$ . Hence,  $K_1 \sim K_2$ . □

Theorems 3.2 and 3.3 fully characterize the partition of  $\mathcal{K}$  by the Cantor-Bendixson characteristic.

### 3.2. Cardinality of the set $\mathcal{K}$

Combining the previous results we obtain the cardinality of  $\mathcal{K}$ .

**Theorem 3.4.** *The set  $\mathcal{K}$ , given by (1. 4), has cardinality  $\aleph_1$ .*

**Proof.** We define the function

$$\begin{aligned} \widetilde{\mathcal{CB}}: \mathcal{K} &\mapsto (\Omega \times (\omega \setminus \{0\})) \cup (0, 0) \\ [K] &\mapsto \widetilde{\mathcal{CB}}([K]) = \mathcal{CB}(K) = (\alpha, p). \end{aligned} \quad (3.12)$$

By Theorem 3.2 and Proposition 3.1, we see that  $\widetilde{\mathcal{CB}}$  is well-defined. Moreover, Corollary 2.1 implies that  $\widetilde{\mathcal{CB}}$  is a surjective function. Furthermore, by Theorem 3.3,  $\widetilde{\mathcal{CB}}$  is an injective function. Then,

$$|\mathcal{K}| = |(\Omega \times (\omega \setminus \{0\})) \cup (0, 0)| = |\Omega \times \omega| = |\Omega| = \aleph_1. \quad \square$$

Last theorem shows that

$$\aleph_0 < \aleph_1 = |\mathcal{K}| \leq 2^{\aleph_0} = \mathfrak{c},$$

where  $\mathfrak{c}$  is the cardinality of  $\mathbb{R}$ .

### 3.3. A “primitive” related to the Cantor-Bendixson derivative of compact subsets of the real line

We end this paper with a last theorem that we can view as a generalization of Theorem 2.1 and Corollary 2.1 given in Section 2. The next result shows that for any compact subset of the reals, there is a primitive-like set associated to its Cantor-Bendixson derivative.

**Theorem 3.5.** *Suppose that  $\alpha \in \Omega$ . Let  $F$  be a compact subset of  $\mathbb{R}$ . Then, there exists a compact set  $\mathcal{F} \subset \mathbb{R}$  such that  $\mathcal{F}^{(\alpha)} = F$ .*

**Proof.** If  $\alpha = 0$ , we define  $\mathcal{F} = F$  and the result holds.

From now on, we suppose that  $\alpha > 0$ . There are two cases. First, if  $F$  is perfect, i.e.,  $F = F'$ , we can take  $\mathcal{F} = F$ , and the result follows.

We now assume that  $F \neq F'$ . Since  $F \setminus F'$  is the set of all isolated points of  $F$ , we have that  $F \setminus F' \neq \emptyset$  is countable. Hence,  $F \setminus F' = \{x_n : n \in I\}$ , where  $\emptyset \neq I \subset \omega$ , and  $x_n \neq x_m$ , for all  $n, m \in I$  with  $n \neq m$ . Furthermore, for all  $n \in I$ , there exists  $r_n \in (0, \frac{1}{n+1})$  such that  $(x_n - r_n, x_n + r_n) \cap F = \{x_n\}$ . By Theorem 2.1, we see that for all  $n \in I$ , there exists  $K_n \in \mathcal{K}$  such that  $K_n \subset (x_n - r_n, x_n]$  and  $K_n^{(\alpha)} = \{x_n\}$ . Since  $((x_n - r_n, x_n])_{n \in I}$  is a pairwise disjoint sequence of intervals, we see that  $(K_n)_{n \in I}$  is a pairwise disjoint sequence in  $\mathcal{K}$ . We now define the set  $\mathcal{F} \subset \mathbb{R}$  given by

$$\mathcal{F} := \biguplus_{n \in I} K_n \cup F. \quad (3.13)$$

**Claim 1.**  $\mathcal{F}$  is a compact subset of  $\mathbb{R}$ .

In fact, let  $(z_k)_{k \in \omega}$  be a sequence in  $\mathcal{F}$  such that  $z_k \rightarrow z \in \mathbb{R}$  when  $k \rightarrow +\infty$ . There are three cases.

(i) If  $\{k \in \omega : z_k \in F\}$  is infinite, there exists a subsequence  $(z_{\phi(k)})_{k \in \omega}$  in  $F$ , where  $\phi : \omega \mapsto \omega$  is a strictly increasing function. Since  $F$  is closed, we conclude that  $z \in F \subset \mathcal{F}$ .

(ii) We now suppose that there exists  $m \in I$  such that  $\{k \in \omega : z_k \in K_m\}$  is infinite. Similarly as in the previous case, we obtain that  $z \in K_m \subset \mathcal{F}$ .

(iii) Finally, we assume that for all  $n \in I$ ,  $\{k \in \omega : z_k \in K_n\}$  is a finite set and  $\{k \in \omega : z_k \in F\}$  is also finite. Thus, there exists a subsequence  $(z_{\psi(k)})_{k \in \omega}$ , where  $\psi : \omega \mapsto \omega$  is a strictly increasing function, and there is also a strictly increasing function  $\sigma : \omega \mapsto I$  such that for all  $k \in \omega$

$$z_{\psi(k)} \in K_{\sigma(k)} \subset (x_{\sigma(k)} - r_{\sigma(k)}, x_{\sigma(k)}]. \quad (3.14)$$

In order to prove the last assertion, we see that there exists  $n_0 \in I$  such that  $\{k \in \omega : z_k \in K_{n_0}\} \neq \emptyset$ . Then, there is  $k_0 \in \omega$  with  $z_{k_0} \in K_{n_0}$ . We thus define  $\psi(0) := k_0$  and  $\sigma(0) := n_0$ . We now get  $n_1 \in I$  with  $n_1 > n_0$  and such that  $\{k \in \omega : z_k \in K_{n_1}, k > k_0\} \neq \emptyset$ . So, there exists  $k_1 \in \omega$  with  $k_1 > k_0$  and such that  $z_{k_1} \in K_{n_1}$ . We define  $\psi(1) := k_1$  and  $\sigma(1) := n_1$ . By continuing this process, functions  $\psi$  and  $\sigma$  are recursively obtained. From (3.14), we have that for all  $k \in \omega$ ,  $|x_{\sigma(k)} - z_{\psi(k)}| < r_{\sigma(k)} < \frac{1}{\sigma(k) + 1}$ . As  $(z_{\psi(k)})_{k \in \omega}$  converges to  $z$ , it follows that  $(x_{\sigma(k)})_{k \in \omega}$  also converges to  $z$ . Since, the elements of the last sequence belong to  $F$ , and  $F$  is closed, we conclude that  $z \in F \subset \mathcal{F}$ .

From (i), (ii), and (iii),  $\mathcal{F}$  is a closed subset of  $\mathbb{R}$ . Moreover, since  $F$  is bounded, there exist  $a, b \in \mathbb{R}$ , with  $a < b$ , such that  $F \subset [a, b]$ . Then,  $\mathcal{F} \subset [a - 1, b]$ , i.e.,  $\mathcal{F}$  is bounded. Hence,  $\mathcal{F}$  is a compact subset of  $\mathbb{R}$ .

**Claim 2.**  $\mathcal{F}^{(\alpha)} = F$ .

Actually, we will show the following more general result: For all countable ordinal number  $\beta \in \Omega$  such that  $\beta \leq \alpha$

$$\mathcal{F}^{(\beta)} = \biguplus_{n \in I} K_n^{(\beta)} \cup F. \quad (3.15)$$

In order to prove (3.15), we proceed by transfinite induction as in Theorem 2.1.

(a) If  $\beta = 0$ , then the result holds immediately.

(b) We now suppose that (3.15) is true for a given  $\beta \in \Omega$  such that  $\beta < \alpha$ . We note that for all  $n \in I$ ,  $K_n^{(\beta+1)} \subset \mathcal{F}^{(\beta+1)}$ . Then,

$$\biguplus_{n \in I} K_n^{(\beta+1)} \subset \mathcal{F}^{(\beta+1)}.$$

Furthermore, by the induction hypothesis,  $F \subset \mathcal{F}^{(\beta)}$ . Then,  $F' \subset \mathcal{F}^{(\beta+1)}$ . Moreover,

$$F \setminus F' = \biguplus_{n \in I} \{x_n\} = \biguplus_{n \in I} K_n^{(\alpha)} \subset \biguplus_{n \in I} K_n^{(\beta+1)} \subset \mathcal{F}^{(\beta+1)}.$$

Hence,

$$\biguplus_{n \in I} K_n^{(\beta+1)} \cup F \subset \mathcal{F}^{(\beta+1)}. \quad (3.16)$$

In order to show the reverse inclusion, we take  $x \in \mathcal{F}^{(\beta+1)}$ . Using the induction hypothesis, we see that

$$x \in \mathcal{F}^{(\beta+1)} = (\mathcal{F}^{(\beta)})' = \left( \biguplus_{n \in I} K_n^{(\beta)} \cup F \right)' = \left( \biguplus_{n \in I} K_n^{(\beta)} \right)' \cup F'.$$

Using now Claim 1, we have that  $\mathcal{F}$  is closed. Then,

$$x \in \mathcal{F}^{(\beta+1)} \subset \mathcal{F}^{(\beta)} = \biguplus_{n \in I} K_n^{(\beta)} \cup F.$$

If  $x \in F$ , there is nothing left to show. On the other hand, if  $x \notin F$ , there exists  $m \in I$  such that  $x \in K_m^{(\beta)} \subset (x_m - r_m, x_m]$ . We now assume, by contradiction, that  $x \notin K_m^{(\beta+1)}$ . Then,  $x$  is an isolated point of  $K_m^{(\beta)}$ .

Since  $x \neq x_m \in F$ , there is  $0 < \varepsilon < \min\{x - x_m + r_m, x_m - x\}$  such that

$$(x - \varepsilon, x + \varepsilon) \cap K_m^{(\beta)} = \{x\}.$$

Moreover, as  $(x - \varepsilon, x + \varepsilon) \subset (x_m - r_m, x_m)$ , we conclude that for all  $n \in I$  with  $n \neq m$ ,

$$(x - \varepsilon, x + \varepsilon) \cap K_n^{(\beta)} = \emptyset.$$

Then,

$$(x - \varepsilon, x + \varepsilon) \cap \biguplus_{n \in I} K_n^{(\beta)} = \{x\}.$$

Therefore,  $x$  is an isolated point of  $\biguplus_{n \in I} K_n^{(\beta)}$ . Since  $x \notin F$ , and  $F$  is closed, we see that  $x \notin F'$ . Hence,  $x \in \left(\biguplus_{n \in I} K_n^{(\beta)}\right)'$ , which is contradictory. In consequence,

$$x \in K_m^{(\beta+1)} \subset \biguplus_{n \in I} K_n^{(\beta+1)}.$$

Thus, summarizing, we can conclude that

$$\mathcal{F}^{(\beta+1)} \subset \biguplus_{n \in I} K_n^{(\beta+1)} \cup F. \quad (3.17)$$

From (3.16) and (3.17), we get

$$\mathcal{F}^{(\beta+1)} = \biguplus_{n \in I} K_n^{(\beta+1)} \cup F.$$

(c) Finally, let  $\gamma \neq 0$  be a limit ordinal such that  $\gamma \leq \alpha$  and we assume that for all ordinal number  $\delta$  such that  $\delta < \gamma$ ,

$$\mathcal{F}^{(\delta)} = \biguplus_{n \in I} K_n^{(\delta)} \cup F. \quad (3.18)$$

Using (3.18), we obtain

$$\begin{aligned} \biguplus_{n \in I} K_n^{(\gamma)} \cup F &= \biguplus_{n \in I} \left( \bigcap_{\delta < \gamma} K_n^{(\delta)} \right) \cup F \\ &\subset \bigcap_{\delta < \gamma} \left( \biguplus_{n \in I} K_n^{(\delta)} \right) \cup F \\ &= \bigcap_{\delta < \gamma} \left( \biguplus_{n \in I} K_n^{(\delta)} \cup F \right) \\ &= \bigcap_{\delta < \gamma} \mathcal{F}^{(\delta)} \\ &= \mathcal{F}^{(\gamma)}. \end{aligned} \quad (3.19)$$

In order to show the other inclusion, we take  $x \in \mathcal{F}^{(\gamma)}$ . Using the induction hypothesis (3.18), we see that

$$\mathcal{F}^{(\gamma)} = \bigcap_{\delta < \gamma} \mathcal{F}^{(\delta)} = \bigcap_{\delta < \gamma} \left( \biguplus_{n \in I} K_n^{(\delta)} \cup F \right).$$

Then, either  $x \in F$  or for all ordinal number  $\delta$  such that  $\delta < \gamma$ , there exists  $n \in I$  such that  $x \in K_n^{(\delta)}$ . If  $x \in F$ , then there is nothing else to be done. If  $x \notin F$ , there is  $N \in I$  such that  $x \in K_N^{(0)} = K_N$ . We now assume, to get a contradiction, that there is an ordinal number  $\delta_0$  with  $\delta_0 < \gamma$  and such that  $x \notin K_N^{(\delta_0)}$ . Since there is  $l \in I$  with  $l \neq N$  such that  $x \in K_l^{(\delta_0)} \subset K_l$ , we obtain a contradiction with the fact that  $K_l \cap K_N = \emptyset$ . Hence, for all ordinal number  $\delta$  such that  $\delta < \gamma$ ,  $x \in K_N^{(\delta)}$ . In consequence,

$$x \in \bigcap_{\delta < \gamma} K_N^{(\delta)} = K_N^{(\gamma)} \subset \biguplus_{n \in I} K_n^{(\gamma)}.$$

Thus,

$$\mathcal{F}^{(\gamma)} \subset \biguplus_{n \in I} K_n^{(\gamma)} \cup F. \quad (3.20)$$

From (3.19) and (3.20), we have that

$$\mathcal{F}^{(\gamma)} = \biguplus_{n \in I} K_n^{(\gamma)} \cup F.$$

By (a), (b) and (c), we obtain (3.15) for all countable ordinal number  $\beta$  such that  $\beta \leq \alpha$ . Finally, using (3.15) with  $\alpha$ , we get

$$\mathcal{F}^{(\alpha)} = \biguplus_{n \in I} K_n^{(\alpha)} \cup F = \biguplus_{n \in I} \{x_n\} \cup F = F,$$

which finishes the proof.  $\square$



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